Optimization using a SAT solver

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2023-03-28

SAT

- Boolean Satisfiability problem: $\mathrm{SAT} : \mathcal{F} \rightarrow \{\mathrm{true}, \mathrm{false}\}$
  deciding if there is an assignment of to the variables of a Boolean formula such that the formula is satisfied (true).
- Consider the formula $\phi = (a \vee b) \wedge (\neg a \vee \neg c)$
  - The assignment $b = \top$ and $c = \bot$ satisfies the formula.
  - $\mathrm{SAT}(\phi) = \mathrm{true}$
- Can be considered a form of mathematical programming, but with a different history and focus.

Satisfiability and validity

- Asking whether

  $$\phi(\vec{x})$$

  is satisfiable is the same as asking whether

  $$\exists \vec{x}. \phi(\vec{x})$$

  is true.

Satisfiability and validity

- In automated theorem proving, it is often more interesting to check validity:

$$\forall x.\phi(x)$$

- Transform to a SAT problem:

$$\forall x.\phi(x) \Leftrightarrow \neg\exists x \neg\phi(x) \quad \text{and check } \mathrm{SAT}(\neg\phi(x))$$

- Special case, optimization (minimization):

$$\forall x(\phi(x) \rightarrow \mathrm{obj}(x) \geq c) \Leftrightarrow \neg\exists x\,(\phi(x) \wedge \mathrm{obj}(x) < c)$$

SAT solvers

- SAT solver: given a formula, find an assignment to the variables, or report that none exist.
- More recently: given a formula, determine its satisfiability and produce either a satisfying assignment or a proof that none exist.
- For a theorem $\forall x. \phi(x)$:
  - $\mathrm{SAT}(\neg\phi(x)) = \top$ corresponds to a counter-example
  - $\mathrm{SAT}(\neg\phi(x)) = \bot$ corresponds to a proof
- In constrast, in OR, infeasibility is often “just a bug”.
  - 4% of instances of MIPLIB2017 are infeasible
  - 53% of instances of the SAT Competition 2022 were unsatisfiable

Conjunctive normal form

- Most SAT solvers take only formulas given in conjunctive normal form

$$(x \vee \neg y \vee \ldots) \wedge (z \vee \neg w \vee \ldots) \wedge \ldots$$

- Variables and negated variables are called *literals*.
- Disjunctions of literals are called *clauses*.

$$x \vee y \vee \neg z \qquad (\Leftrightarrow x + y + (1 - z) \geq 1)$$

- CNF is a conjunction of clauses.
- By adding some variables, we can transform any propositional formula to CNF (with worst-case linear increase in size).

CNF encoding

- Good formulations (*encodings*) are still very important.
    - Cardinality constraints are almost a sub-field in itself

$$x + y + \neg z + \ldots \leq k$$

    - Pseudo-Boolean constraints (almost binary ILP a.k.a $0, 1$-programs)

$$a_1 x_1 + a_2(\neg x_2) + \ldots \leq k$$

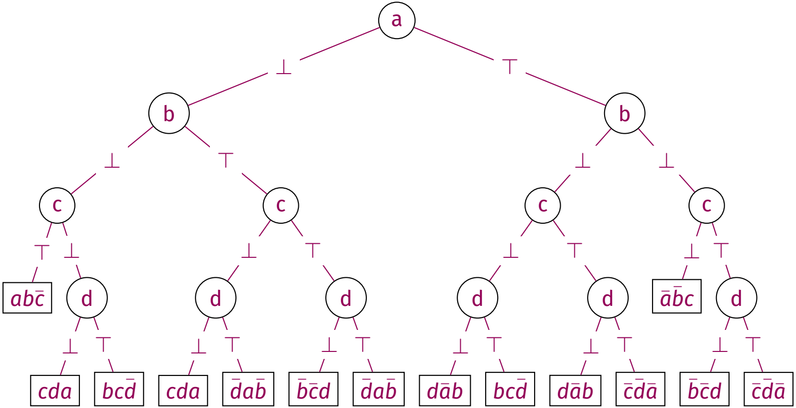
- We are not worrying about this today
- $x + y \leq 1$ can be encoded $\neg x \vee \neg y$
- $x + y + z \leq 1 \Rightarrow \quad (\neg x \vee \neg y) \wedge (\neg x \vee \neg z) \wedge (\neg y \vee \neg z)$

An example computation

Consider the formula

$$
\begin{aligned}
\phi(a, b, c, d) = & (a \vee b \vee \neg c) \wedge \\
& (b \vee c \vee \neg d) \wedge \\
& (c \vee d \vee a) \wedge \\
& (d \vee \neg a \vee b) \wedge \\
& (\neg a \vee \neg b \vee c) \wedge \\
& (\neg b \vee \neg c \vee d) \wedge \\
& (\neg c \vee \neg d \vee \neg a) \wedge \\
& (\neg d \vee a \vee \neg b)
\end{aligned}
$$

Tree search (case splitting)

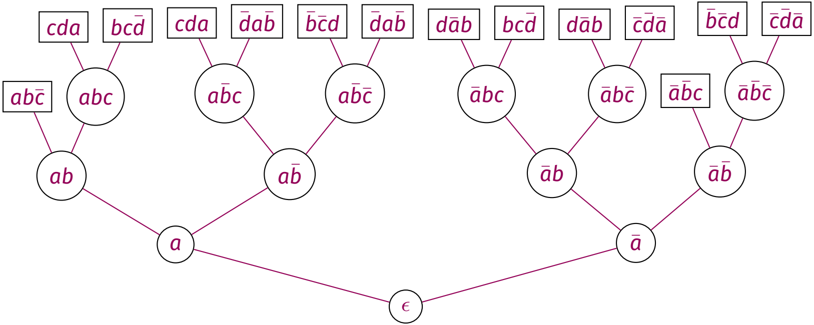


How to prove something

- Implicitly by algoritm: the algorithm is correct for all formulas. It returned UNSAT for this formula, so the formula is unsatisfiable.
  - What if the solver is incorrect?
  - Maybe an explicit proof is interesting in itself?
- ... or **explicitly** by defining a proof structure.
- Need a set of *proof rules*.
- For propositional logic, one is enough:

$$\frac{x \vee \phi \qquad \neg x \vee \psi}{\phi \vee \psi} \text{ Resolution}$$

Proof tree: resolution steps



Conflict-driven clause learning

- Conflict-driven clause learning (**CDCL**) is an algorithm that attacks the problem from both sides: search and proof.
- Ingredients:
  - Heuristic assignment
  - Unit propagation
  - Conflict analysis $\rightarrow$ clause learning

Knuth’s formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

Trail (level=0):

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

Trail (level=1):

$\boxed{\neg a}$

Guessing $\neg a$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

Trail (level=2):

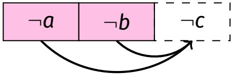
$\neg a$	$\neg b$

Guessing $\neg b$

Knuth’s formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

Trail (level=2):



Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

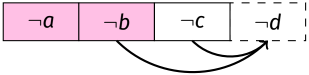
$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

Trail (level=2):



Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Conflict clause: $a \lor c \lor d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Conflict clause: $a \lor c \lor d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Conflict clause: $a \lor b \lor c$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$

$b \vee c \vee \neg d$

$c \vee d \vee a$

$d \vee \neg a \vee b$

$\neg a \vee \neg b \vee c$

$\neg b \vee \neg c \vee d$

$\neg c \vee \neg d \vee \neg a$

$\neg d \vee a \vee \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Conflict clause: $a \vee b \vee c$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Conflict clause: $a \lor b$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

Trail (level=2):

Backtrack to here

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Learnt new clause: $a \lor b$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=2):

$\neg a$	$\neg b$	$\neg c$	$\neg d$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):

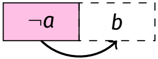
$\boxed{\neg a}$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):



Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$

$b \vee c \vee \neg d$

$c \vee d \vee a$

$d \vee \neg a \vee b$

$\neg a \vee \neg b \vee c$

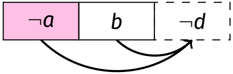
$\neg b \vee \neg c \vee d$

$\neg c \vee \neg d \vee \neg a$

$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):



Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

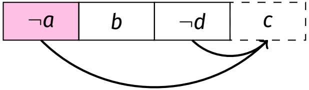
$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):



Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $\neg b \vee d \vee \neg c$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $\neg b \lor d \lor \neg c$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $a \lor \neg b \lor d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $a \lor \neg b \lor d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $a \lor \neg b$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: $a \vee \neg b$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$

Trail (level=1):

$\neg a$	b	$\neg d$	c

Conflict clause: a

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$

Trail (level=1):
Backtrack to here

$\neg a$	b	$\neg d$	c

Learnt new clause: a

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

$\neg a$	b	$\neg d$	c

Knuth’s formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=0):

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=0):

a

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=0):

$\boxed{a}$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

a	$\neg b$

Guessing $\neg b$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$
a

Trail (level=1):

a	$\neg b$	d

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$
a

Trail (level=1):

a	$\neg b$	d

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

$a \lor b$

a

Trail (level=1):

a	$\neg b$	d	c

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

a	$\neg b$	d	c

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

a	$\neg b$	d	c

Conflict clause: $\neg a \lor \neg d \lor \neg c$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$

$b \vee c \vee \neg d$

$c \vee d \vee a$

$d \vee \neg a \vee b$

$\neg a \vee \neg b \vee c$

$\neg b \vee \neg c \vee d$

$\neg c \vee \neg d \vee \neg a$

$\neg d \vee a \vee \neg b$

$a \vee b$

a

Trail (level=1):

a	$\neg b$	d	c

Conflict clause: $\neg a \vee \neg d \vee \neg c$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$
a

Trail (level=1):

a	$\neg b$	d	c

Conflict clause: $\neg a \vee b \vee \neg d$

Knuth's formula (6): solver trace

$a \vee b \vee \neg c$

$b \vee c \vee \neg d$

$c \vee d \vee a$

$d \vee \neg a \vee b$

$\neg a \vee \neg b \vee c$

$\neg b \vee \neg c \vee d$

$\neg c \vee \neg d \vee \neg a$

$\neg d \vee a \vee \neg b$

$a \vee b$

a

Trail (level=1):

a	$\neg b$	d	c

Conflict clause: $\neg a \vee b \vee \neg d$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

a	$\neg b$	d	c

Conflict clause: $\neg a \lor b$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

Trail (level=1):

Backtrack to here

a	$\neg b$	d	c

Learnt new clause: $\neg a \lor b$

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

$\neg d \lor a \lor \neg b$

$a \lor b$

a

$\neg a \lor b$

Trail (level=1):

a	$\neg b$	d	c

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

$\neg a \lor b$

Trail (level=0):

$\boxed{a}$

Knuth’s formula (6): solver trace

$a \vee b \vee \neg c$

$b \vee c \vee \neg d$

$c \vee d \vee a$

$d \vee \neg a \vee b$

$\neg a \vee \neg b \vee c$

$\neg b \vee \neg c \vee d$

$\neg c \vee \neg d \vee \neg a$

$\neg d \vee a \vee \neg b$

$a \vee b$

a

$\neg a \vee b$

Trail (level=0):

$\boxed{a}$ b

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

$\neg a \lor b$

Trail (level=0):

a	b

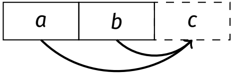
Knuth's formula (6): solver trace

$a \vee b \vee \neg c$
$b \vee c \vee \neg d$
$c \vee d \vee a$
$d \vee \neg a \vee b$
$\neg a \vee \neg b \vee c$
$\neg b \vee \neg c \vee d$
$\neg c \vee \neg d \vee \neg a$
$\neg d \vee a \vee \neg b$

$a \vee b$
a

$\neg a \vee b$

Trail (level=0):



Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

$\neg a \lor b$

Trail (level=0):

a	b	c

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$

$b \lor c \lor \neg d$

$c \lor d \lor a$

$d \lor \neg a \lor b$

$\neg a \lor \neg b \lor c$

$\neg b \lor \neg c \lor d$

$\neg c \lor \neg d \lor \neg a$

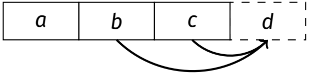
$\neg d \lor a \lor \neg b$

$a \lor b$

a

$\neg a \lor b$

Trail (level=0):



Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

$\neg a \lor b$

Trail (level=0):

a	b	c	d

Knuth's formula (6): solver trace

$a \lor b \lor \neg c$
$b \lor c \lor \neg d$
$c \lor d \lor a$
$d \lor \neg a \lor b$
$\neg a \lor \neg b \lor c$
$\neg b \lor \neg c \lor d$
$\neg c \lor \neg d \lor \neg a$
$\neg d \lor a \lor \neg b$

$a \lor b$
a

$\neg a \lor b$

Trail (level=0):

a	b	c	d

Conflict clause: $\neg c \lor \neg d \lor \neg a$

Conflict on decision level zero. UNSAT!

CDCL proofs

The learnt clauses, in order, are a proof in the following sense:

- Take the first learnt clause, and assert its negation:
  - $x \vee y \quad \Rightarrow \quad \neg x \wedge \neg y$
  - A contradiction is derived using unit propagation only
- Consider the clause now a part of the formula (it is implied)
- After the last learnt clause has been added to the formula, a contradiction can be found using unit propagation.
- The formula, including the *original* clauses and some **implied** clauses, can be shown inconsistent by unit propagation $\Rightarrow$ the original formula is *UNSAT*.

Pythagorean triplets

MailOnline Science & Tech

World's longest maths proof: Solution to a 30-year-old problem would take 10 BILLION years to read - all for a prize of just $100

- **In 1980s American mathematician offered a prize to solve a maths problem**
- **Problem involves assigning sides of a right-angled triangle two colours**
- **Now 30 years later a team have solved it, with the help of a supercomputer**
- **But the solution is 200 terabytes of data, which is too large to read**

By ABIGAIL BEALL FOR MAILONLINE

PUBLISHED: 16:13 GMT, 11 July 2016 | **UPDATED:** 18:44 GMT, 11 July 2016

In the 1980s an American mathematician named Robert Graham offered a prize of $100 (£77) to anyone able to solve a brain-teaser that he could not solve himself.

Optimization

Finally: optimization

Optimization

- A few different problem definitions
  - **MaxSAT**: satisfy the maximum number of clauses of a SAT problem
  - (*weighted*) (*partial*) MaxSAT
  - **Pseudo-boolean** Optimization (0,1 ILP)
  - **Weighted Boolean** optimization
- The differences are not too interesting

$$\begin{aligned} \text{min} \quad & \Sigma w_i x_i \\ \text{s.t.} \quad & \phi(x_1, x_2, \ldots) \end{aligned}$$

Main MaxSAT approaches

- Branch and bound
- Stochastic local search
- SAT-based algorithms
  - Iterative SAT solving
  - **Core-guided** algorithms

Iterative SAT solving

- Find some way to translate the constraint $(\text{obj}(x) \leq k) = (\Sigma w_i x_i \leq k)$ to propositional logic
- Solve a sequence of SAT problems to find a k value s.t.
  - $SAT(\phi(x) \wedge \text{obj}(x) \leq k) = \text{false}$, and
  - $SAT(\phi(x) \wedge \text{obj}(x) \leq k + 1) = \text{true}$.
- Drawback: the objective can become a large formula

Core-guided algorithms

- (Assume $w_i \in \{0,1\}$, and let S be the set of i's for which $w_i = 1$.)
- The special case $k = 0$ ($\Sigma_{i \in S} x_i \leq 0$) doesn't require any large encodings:

$$\phi(x) \wedge \bigwedge_{a \in A_0} a$$

$$A_0 = \{\neg x_i \mid i \in S\}$$

$$A_0 = \{x_i \leq 0 \mid i \in S\}$$

- If $\mathrm{SAT}(\ldots) = \mathrm{true}$ – good, the optimum was zero.

Core-guided algorithms

- If UNSAT and the proof contains at least one clause

$$\neg C \subseteq A_0$$

- Example $A_0 = \{\neg x, \neg y, \neg z\}$ and $C = x \vee y$.
- We know that $obj(x) \geq 1$
- Can produce a new formula representing **all** $obj(x) \leq 1$ solutions

$$\phi(x) \wedge \bigwedge_{a \in A_1} a$$

$$\begin{aligned} A_1 = {} & (A_0 \setminus \neg C) \cup \\ & \{\Sigma x_i \leq k+1 \mid \Sigma x_i \leq k \in \neg C\} \cup \\ & \{\Sigma_{x \in C}\, x \leq 1\} \end{aligned}$$

Assumptions

- We can make sure that $C \subseteq A$ by setting $x \in A$ as the first decisions.
- Call this **assumptions**
- Stop when a conflict at decision level $\leq |A|$
- The *last* learnt clause C will be a subset of A.

Example:

- Schedule three activities **A**, **B**, **C** with a resource limit of *2*.

Activity	Duration	Resources	Earliest	Latest
A	1	1	0	3
B	1	2	0	3
C	2	1	0	2

- Optimize sum of starting times

$$\min \Sigma s_i$$

CNF encoding of start time

- Unary encoding of start time of activities.
  - Boolean variables represent lower bounds:
    - $t_A^x := (s_A \geq x) \qquad \neg t_A^x := (s_A < x)$
    - $t_A^1, t_A^2, t_A^3, t_B^1, t_B^2, t_B^3, t_C^1, t_C^2$
  - Constraints:
    - $t_A^3 \Rightarrow t_A^2, t_A^3 \Rightarrow t_A^2, t_B^3 \Rightarrow t_B^2, t_B^3 \Rightarrow t_B^2, t_C^2 \Rightarrow t_C^1$
  - Examples:
    - $[t_A^1, t_A^2, t_A^3] = [0, 0, 0] \Rightarrow s_A = 0$
    - $[t_A^1, t_A^2, t_A^3] = [1, 0, 0] \Rightarrow s_A = 1$
    - $[t_A^1, t_A^2, t_A^3] = [1, 1, 0] \Rightarrow s_A = 2$
    - $[t_A^1, t_A^2, t_A^3] = [1, 1, 1] \Rightarrow s_A = 3$

CNF encoding of resource constraints

- Resource limit is 2
  - $\Rightarrow$ activity **B** cannot run at the same time as **A** or **C**.
- Convert this to clauses over t's:
  - **B** start at time 0:
    - $t_B^1 \vee t_A^1$
    - $t_B^1 \vee t_C^1$
  - **B** start at time 1:
    - $t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
    - $t_B^2 \vee \neg t_B^1 \vee t_C^2$
  - **B** start at time 2:
    - $t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
    - $t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
  - **B** start at time 3:
    - $\neg t_B^3 \vee \neg t_A^3$
    - $\neg t_B^3 \vee \neg t_C^2$

Assumptions

- UB=0 solutions:

$$\begin{aligned} A_0 &= \{(s_A \leq 0), (s_B \leq 0), (s_C \leq 0)\} \\ &= \{\neg t_A^1, \neg t_B^1, \neg t_C^1\} \end{aligned}$$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=0):

Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

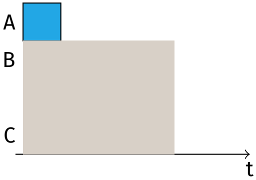
Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$

Using assumption $\neg t_A^1$



Scheduling example: solver trace

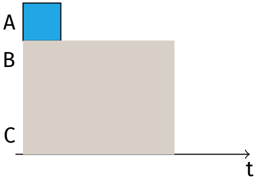
$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$



Scheduling example: solver trace

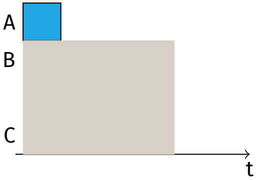
$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$



Scheduling example: solver trace

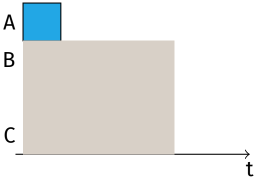
$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$



Scheduling example: solver trace

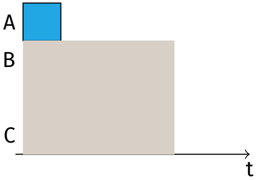
$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$



Scheduling example: solver trace

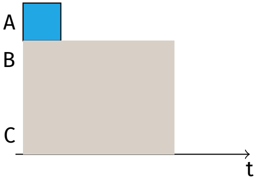
$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

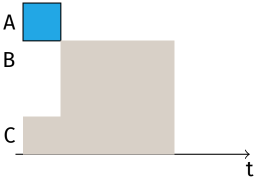
$\neg t_B^3 \lor \neg t_C^2$

Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

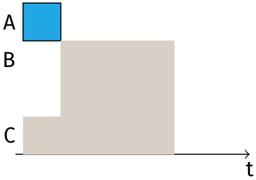
Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1

Conflict clause: $\neg t_B^1 \lor t_B^1$



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

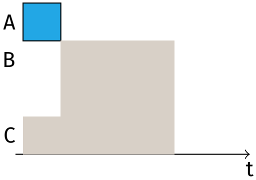
Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1

Conflict clause: $\neg t_B^1 \vee t_B^1$



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

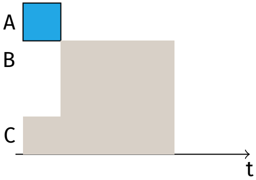
Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1

Conflict clause: $t_A^1 \vee t_B^1$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

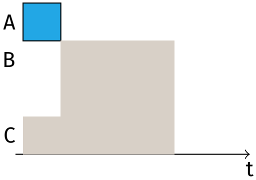
Assumptions:

$\neg t_A^1$	$\neg t_B^1$	$\neg t_C^1$

Trail (level=1):

$\neg t_A^1$	$\neg t_A^2$	$\neg t_A^3$	t_B^1

Conflict clause: $t_A^1 \lor t_B^1$



UB=1

$$
\begin{aligned}
A_0 &= \left\{\neg t_A^1, \neg t_B^1, \neg t_C^1\right\}, \qquad C = t_A^1 \vee t_B^1 \\
A_1 &= (A_0 \setminus \neg C) \cup \\
&\quad \{\Sigma x_i \leq k+1 \mid \Sigma x_i \leq k \in \neg C\} \cup \\
&\quad \{\Sigma_{x \in C}\, x \leq 1\} \\
&= \left\{\neg t_C^1\right\} \cup \{s_A \leq 1, s_B \leq 1\} \cup \left\{t_A^1 + t_B^1 \leq 1\right\} \\
&= \left\{\neg t_C^1, \neg t_A^2, \neg t_B^2, \neg v_{A1}^{B1}\right\}
\end{aligned}
$$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=0):

Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

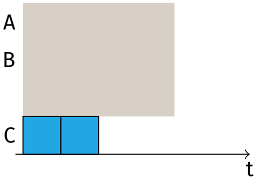
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$

Using assumption $\neg t_C^1$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

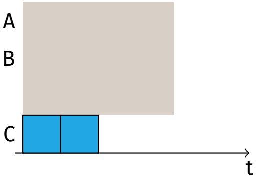
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

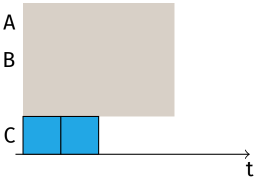
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

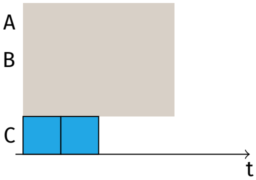
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

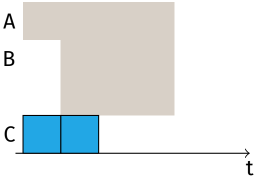
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

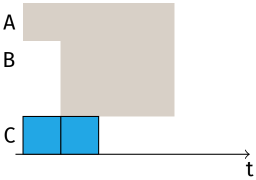
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2



Scheduling example: solver trace

$$\neg t_A^2 \lor t_A^1$$
$$\neg t_A^3 \lor t_A^2$$
$$\neg t_B^2 \lor t_B^1$$
$$\neg t_B^3 \lor t_B^2$$
$$\neg t_C^2 \lor t_C^1$$
$$t_B^1 \lor t_A^1$$
$$t_B^1 \lor t_C^1$$
$$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$$
$$t_B^2 \lor \neg t_B^1 \lor t_C^2$$
$$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$$
$$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$$
$$\neg t_B^3 \lor \neg t_A^3$$
$$\neg t_B^3 \lor \neg t_C^2$$

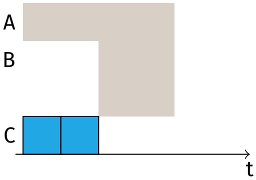
$$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=1):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

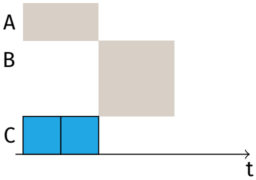
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$

Using assumption $\neg t_A^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

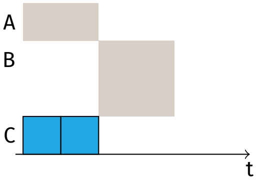
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

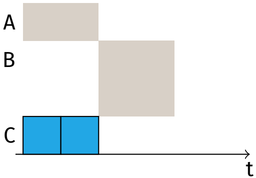
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

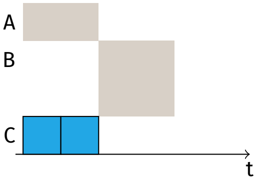
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $\neg t_B^2 \vee t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

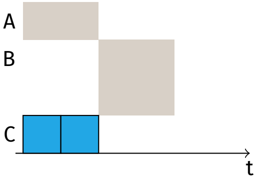
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $\neg t_B^2 \lor t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

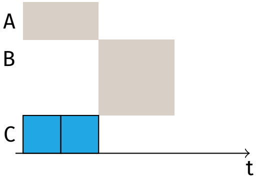
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^2 \lor \neg t_B^1 \lor t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$

$\neg t_A^3 \vee t_A^2$

$\neg t_B^2 \vee t_B^1$

$\neg t_B^3 \vee t_B^2$

$\neg t_C^2 \vee t_C^1$

$t_B^1 \vee t_A^1$

$t_B^1 \vee t_C^1$

$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$

$t_B^2 \vee \neg t_B^1 \vee t_C^2$

$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$

$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$

$\neg t_B^3 \vee \neg t_A^3$

$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

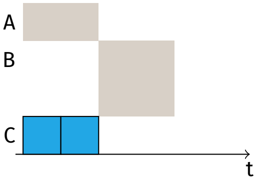
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^2 \vee \neg t_B^1 \vee t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

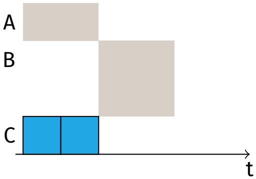
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^1 \lor \neg t_B^1 \lor t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

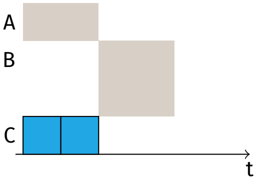
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^1 \lor \neg t_B^1 \lor t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

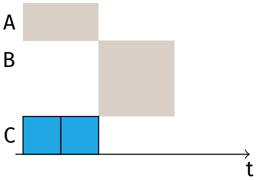
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^1 \lor t_B^2$



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

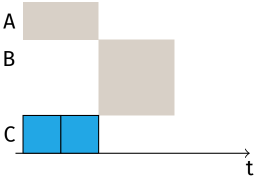
Assumptions:

$\neg t_C^1$	$\neg t_A^2$	$\neg t_B^2$	$\neg v_{A1}^{B1}$

Trail (level=2):

$\neg t_C^1$	$\neg t_C^2$	t_B^1	t_B^2	$\neg t_A^2$	$\neg t_A^3$

Conflict clause: $t_C^1 \vee t_B^2$



UB=2

$$
\begin{aligned}
A_1 &= \left\{ \neg t_C^1, \neg t_A^2, \neg t_B^2, \neg v_{A1}^{B1} \right\}, \qquad C = t_C^1 \vee t_B^2 \\
A_2 &= \left\{ \neg t_A^2, \neg v_{A1}^{B1} \right\} \cup \{ s_C \leq 1, s_B \leq 2 \} \cup \left\{ t_C^1 + t_B^2 \leq 1 \right\} \\
&= \left\{ \neg t_A^2, \neg v_{A1}^{B1}, \neg t_C^2, \neg t_B^3, \neg v_{C1}^{B2} \right\}
\end{aligned}
$$

Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$
$t_C^1 \lor t_B^2$
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=0):

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=1):

$\neg t_A^2$

Using assumption $\neg t_A^2$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=1):

$\neg t_A^2$	$\neg t_A^3$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=1):

$\neg t_A^2$	$\neg t_A^3$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=2):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$

Using assumption $\neg v_{A1}^{B1}$

Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

$t_C^1 \lor t_B^2$

$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=3):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$

Using assumption $\neg t_B^3$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=4):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$

Using assumption $\neg t_C^2$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=5):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Using assumption $\neg v_{B2}^{C1}$

Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$

$\neg t_A^3 \vee t_A^2$

$\neg t_B^2 \vee t_B^1$

$\neg t_B^3 \vee t_B^2$

$\neg t_C^2 \vee t_C^1$

$t_B^1 \vee t_A^1$

$t_B^1 \vee t_C^1$

$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$

$t_B^2 \vee \neg t_B^1 \vee t_C^2$

$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$

$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$

$\neg t_B^3 \vee \neg t_A^3$

$\neg t_B^3 \vee \neg t_C^2$

$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$

$t_C^1 \vee t_B^2$

$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

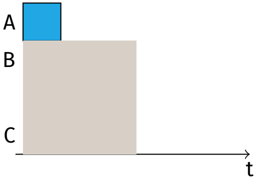
Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$

Guessing $\neg t_A^1$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

$t_C^1 \lor t_B^2$

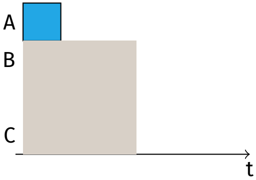
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

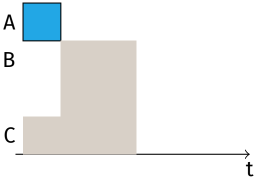
$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1



Scheduling example: solver trace

$\neg t_A^2 \vee t_A^1$
$\neg t_A^3 \vee t_A^2$
$\neg t_B^2 \vee t_B^1$
$\neg t_B^3 \vee t_B^2$
$\neg t_C^2 \vee t_C^1$
$t_B^1 \vee t_A^1$
$t_B^1 \vee t_C^1$
$t_B^2 \vee \neg t_B^1 \vee \neg t_A^1 \vee t_A^2$
$t_B^2 \vee \neg t_B^1 \vee t_C^2$
$t_B^3 \vee \neg t_B^2 \vee \neg t_A^2 \vee t_A^3$
$t_B^3 \vee \neg t_B^2 \vee \neg t_C^1$
$\neg t_B^3 \vee \neg t_A^3$
$\neg t_B^3 \vee \neg t_C^2$

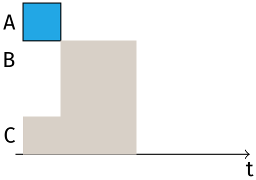
$v_{A1}^{B1} \vee \neg t_A^1 \vee \neg t_B^1$
$t_C^1 \vee t_B^2$
$v_{B2}^{C1} \vee \neg t_B^2 \vee \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1	t_B^2



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$

$\neg t_A^3 \lor t_A^2$

$\neg t_B^2 \lor t_B^1$

$\neg t_B^3 \lor t_B^2$

$\neg t_C^2 \lor t_C^1$

$t_B^1 \lor t_A^1$

$t_B^1 \lor t_C^1$

$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$

$t_B^2 \lor \neg t_B^1 \lor t_C^2$

$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$

$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$

$\neg t_B^3 \lor \neg t_A^3$

$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$

$t_C^1 \lor t_B^2$

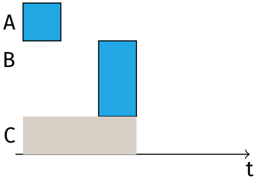
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1	t_B^2



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

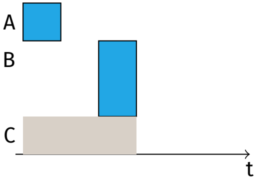
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$
$t_C^1 \lor t_B^2$
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1	t_B^2	$\neg t_C^1$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

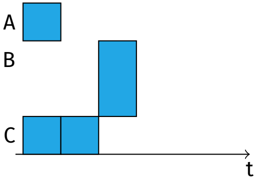
$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$
$t_C^1 \lor t_B^2$
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1	t_B^2	$\neg t_C^1$



Scheduling example: solver trace

$\neg t_A^2 \lor t_A^1$
$\neg t_A^3 \lor t_A^2$
$\neg t_B^2 \lor t_B^1$
$\neg t_B^3 \lor t_B^2$
$\neg t_C^2 \lor t_C^1$
$t_B^1 \lor t_A^1$
$t_B^1 \lor t_C^1$
$t_B^2 \lor \neg t_B^1 \lor \neg t_A^1 \lor t_A^2$
$t_B^2 \lor \neg t_B^1 \lor t_C^2$
$t_B^3 \lor \neg t_B^2 \lor \neg t_A^2 \lor t_A^3$
$t_B^3 \lor \neg t_B^2 \lor \neg t_C^1$
$\neg t_B^3 \lor \neg t_A^3$
$\neg t_B^3 \lor \neg t_C^2$

$v_{A1}^{B1} \lor \neg t_A^1 \lor \neg t_B^1$
$t_C^1 \lor t_B^2$
$v_{B2}^{C1} \lor \neg t_B^2 \lor \neg t_C^1$

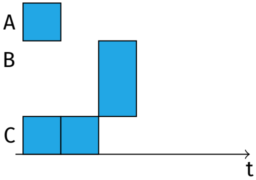
Assumptions:

$\neg t_A^2$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$

Trail (level=6):

$\neg t_A^2$	$\neg t_A^3$	$\neg v_{A1}^{B1}$	$\neg t_B^3$	$\neg t_C^2$	$\neg v_{B2}^{C1}$	$\neg t_A^1$	t_B^1	t_B^2	$\neg t_C^1$

SAT!



Conclusions

- We formulated the problem using infeasibility for useful information.
- No explicitly encoded/formulated objective Σs_a (only "lazily").
- The *cores* are valid constraints/cuts, could be used in other algorithms?
- Problem formulation can be built incrementally: adding new variables and constraints preserves:
  - Cores and lower bounds
  - Learnt clauses
  - Adaptive branching/value heuristics
- It's not the whole explicit UNSAT proof that's interesting (we used only the resulting *cores*), but getting new perspectives on how to optimize something.