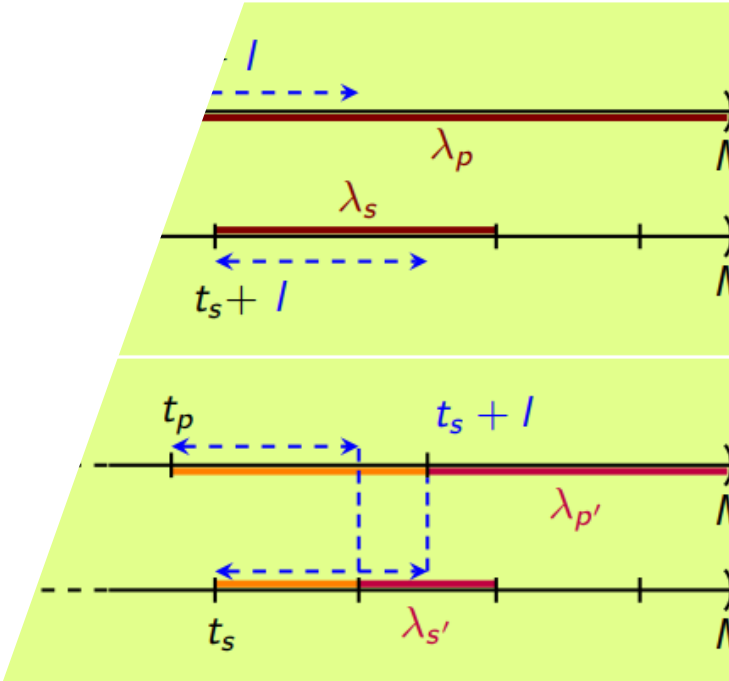
Dynamic time discretization for train scheduling

Bjørnar Luteberget

19 September 2025



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Havbasseng

MiNaLab

Fermentering

Nullutslippsbygg

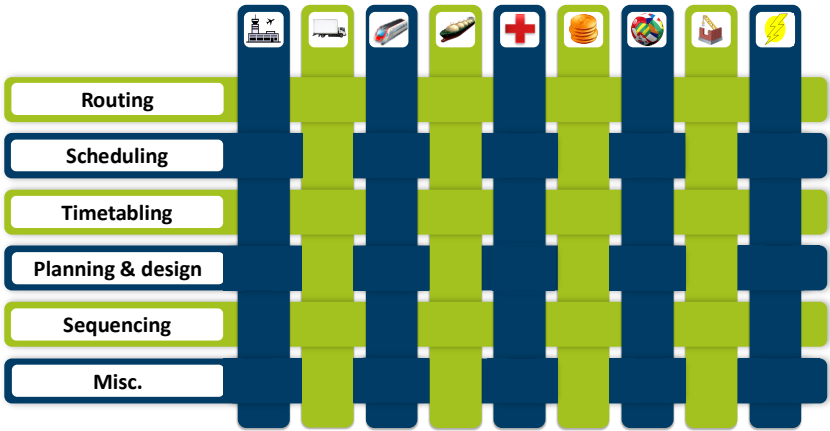
CO_2-lab

Fullskala havbrukslokalitet

Elektroteknisk

Produksjonsprosesser

SINTEF Optimization – Application areas



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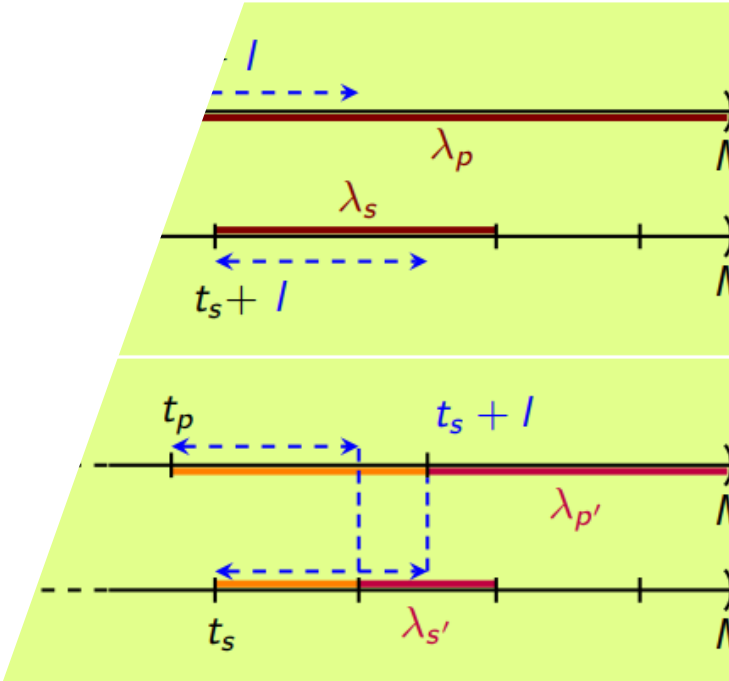

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Background knowledge

This is a **technical talk** on mathematical modeling of scheduling problems.
Assuming that many are familiar with mixed-integer linear programming (MILP):

$$
\begin{aligned}
\text{minimize} \quad & c_1 x_1 + c_2 x_2 + \dots \\
\text{subject to} \quad & a_{1,1} x_1 + a_{1,2} x_2 + \dots \leq b_1 \\
& a_{2,1} x_1 + a_{2,2} x_2 + \dots \leq b_2 \\
& \dots \leq \dots \\
& x_i \in \mathbb{Z} \qquad \forall x_i \in I \subseteq X
\end{aligned}
$$

… and **scheduling problems** are mathematical problems concerning the optimal timing and resource allocation for a set of tasks.

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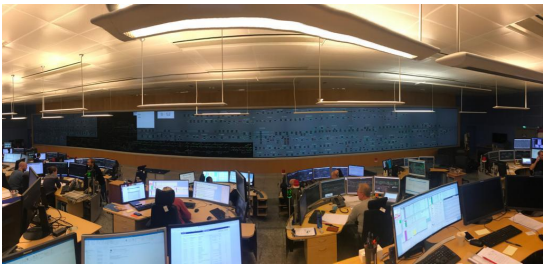
Summary: DDD for train scheduling

- Schedule optimization often uses **time discretization**, but this has **severe drawbacks for train scheduling**.
- We have developed a **Dynamic discretization discovery (DDD)** that can overcome some of these drawbacks.
- We have tested the DDD on a simplified train dispatching problem.
- For competitive performance, the mathematical solver also needs to work dynamically. A **MaxSAT** algorithm outperforms MILP solvers on some objectives.

Joint work with *Anna Livia Croella*, *Carlo Mannino*, and *Paolo Ventura*.
Nominated for INFORMS RAS Student Paper Award 2022.
Journal paper published in Computers & Operations Research.

Timetabling and real-time dispatching

A timetable at Oslo central

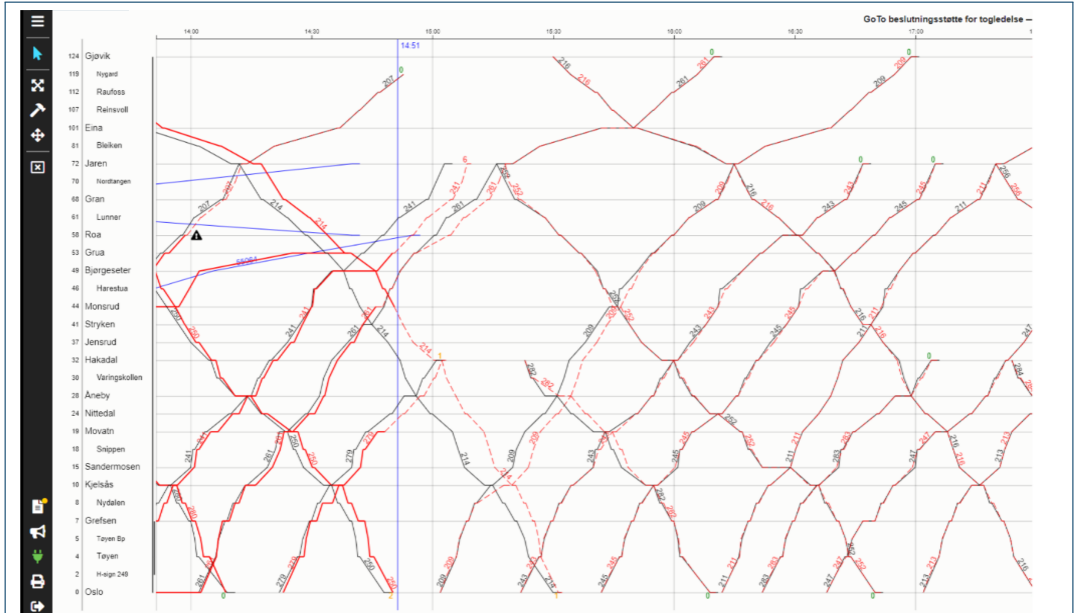


Dispatchers at Oslo control centre

Train re-scheduling

Trains run according to carefully pre-planned timetables,
but **delays and deviations** happen!

- **Timetables** may become **infeasible** and parts of the network may become unavailable for inbound trains
- The original schedule must be adjusted in real-time to mitigate the **impact** on the overall traffic.
- One aim to restore a feasible situation while **minimizing** some measure of the **deviation** of the actual schedule from the official timetable.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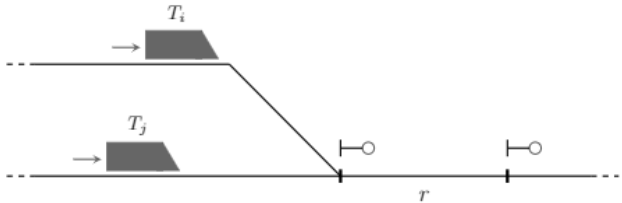
124 Gjøvik
119 Nygard
112 Raufoss
107 Reinsvoll
101 Eina
81 Bleiken
72 Jaren
70 Nordtangen
68 Gran
61 Lunner
58 Roa
53 Grua
49 Bjørgeseter
46 Harestua
44 Monsrud
41 Stryken
37 Jensrud
32 Hakadal
30 Varingskollen
28 Åneby
24 Nittedal
19 Movatn
18 Snippen
15 Sandermosen
10 Kjelsås
8 Nydalen
7 Grefsen
5 Tøyen Bp
4 Tøyen
2 H-sign 249
0 Oslo

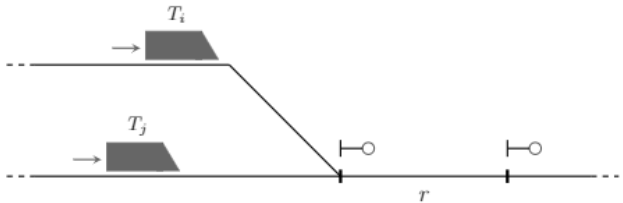
A simplified train re-scheduling problem



- Trains travel on fixed routes through tracks and stations.
- Trains spend a fixed amount of time to traverse a track.
- Station capacities and routing are ignored.

- Extension to **variable** travel times is straight-forward, **station capacity** is easy, general **routing** is possible.

A simplified train re-scheduling problem



- $t_i^{s2} - t_i^{s1} \geq l$
- $t_j^{s2} - t_j^{s1} \geq l$
- $t_j^{s2} - t_i^{s1} \geq 0 \bigvee t_j^{s1} - t_i^{s2} \geq 0$

MILP formulations

Two classes of MILP models are adopted in the literature:

- **big-M formulations** $\Rightarrow$ continuous time variables t_{ir}

$$t_{ir} - t_{jr} + M(1 - y_r^{ij}) \geq l_r^{ij}$$

$$t_{jr} - t_{ir} + M y_r^{ij} \geq l_r^{ji}$$

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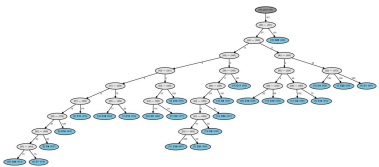
- **Time-Indexed formulations** $\Rightarrow$ discrete time variables x_{ir}^p

$$x_{ir}^p + x_{jr}^q \leq 1$$

Classical MILP formulations drawbacks

big-M formulations

- Poor bounds
- Large branching trees



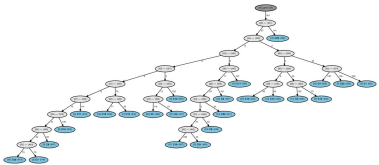
Classical MILP formulations drawbacks

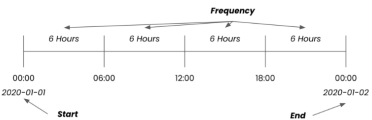
big-M formulations

- Poor bounds
- Large branching trees

TI formulations

- Oversize
- Bad approximation





We introduce a new TI based formulation:
the Interval Assignment Problem (IAP)

A novel paradigm

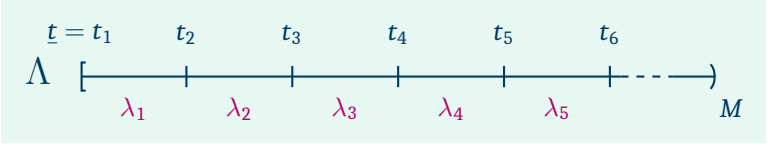
- E. He, N. Boland, G. Nemhauser, and M. Savelsbergh. **A dynamic discretization discovery algorithm for the minimum duration time-dependent shortest path problem**,2018
- Y. He, F. Lehuédé, and O. Péton. **A dynamic discretization approach to the integrated service network design and vehicle routing problem** In VeRoLog 2019 : seventh annual workshop of the EURO Working Group on Vehicle Routing and Logistics Optimization, Sevilla, Spain, June 2019.
- D. M. Vu, M. Hewitt, N. Boland, and M. Savelsbergh. **Dynamic discretization discovery for solving the time-dependent traveling salesman problem with time windows** Transportation Science, 2020
- Y. O. Scherr, M. Hewitt, B. A. N. Saavedra, and D. C. Mattfeld. **Dynamic discretization discovery for the service network design problem with mixed autonomous fleets.** Transportation Research Part B: Methodological, 2020
- L. Marshall, N. Boland, M. Savelsbergh, and M. Hewitt. **Interval-based dynamic discretization discovery for solving the continuous-time service network design problem** Transportation Science, 2021.

The DDD consists in solving a sequence of models with both a **fine discretization** & **limited size**.

Classical TI formulation

For each train and each track segment

$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a partition of the time horizon $[\underline{t}, M)$
such that $\lambda_p = [t_p, t_{p+1})$



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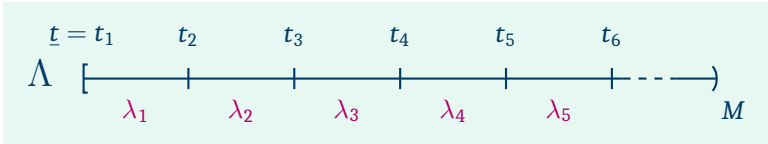
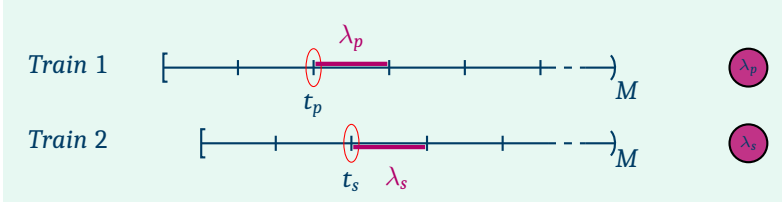


$$x_p = \begin{cases} 1 & \text{if train enters the track segment} \\ & \text{at the beginning } t_p \text{ of the interval } \lambda_p \\ 0 & \text{otherwise} \end{cases}$$

TI-Incompatibility

Given two trains traversing the same track segment, two intervals $\lambda_p = [t_p, t_{p+1})$ and $\lambda_s = [t_s, t_{s+1})$ are said TI-incompatible if

$$[t_p, t_p + l) \cap [t_s, t_s + l) \neq \emptyset$$



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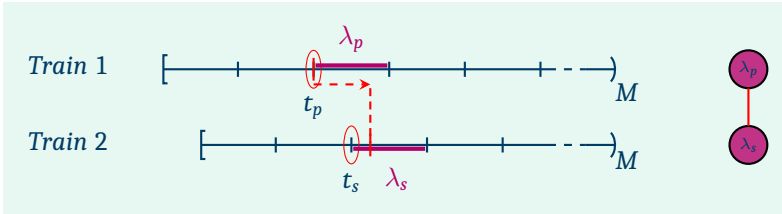

$$x_p + x_s \leq 1 \quad \forall\ \lambda_p \text{ TI-incompatible with } \lambda_s$$

TI formulation

Given a set of partitions $\Lambda = \{\Lambda^{ir} : i \in I, r \in R_i\}$,
we want to find x^*, the incidence vector of a set of non-TI-incompatible intervals of minimum cost $\bar{c}(x^*)$.

$$\min \quad \sum_{i \in I} \sum_{r \in R_i} \sum_{\lambda_p \in \Lambda^{ir}} \bar{c}_p \cdot (x_p)$$

$$s.t.$$

$$(1) \quad \sum_{\lambda_p \in \Lambda^{ir}} x_p = 1, \qquad i \in I,\ r \in R_i$$

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$$x_p \in \{0, 1\} \qquad i \in I,\ r \in R_i,\ \lambda_p \in \Lambda^{ir}$$

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We can obtain a schedule $t^* = \Phi(x^*)$,
if x^* is optimal then t^* is optimal for the TRP

Interval Assignment Problem (IAP)

For each train and each track segment

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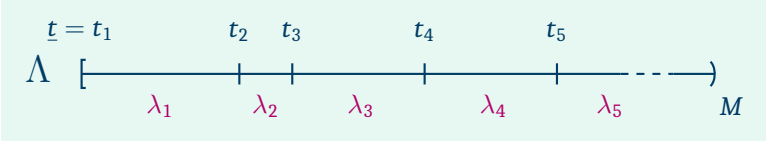

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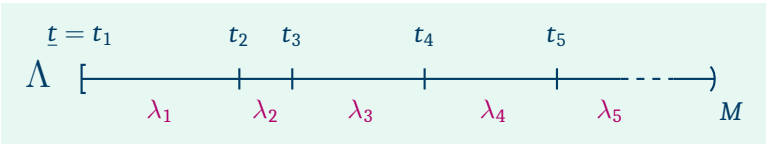
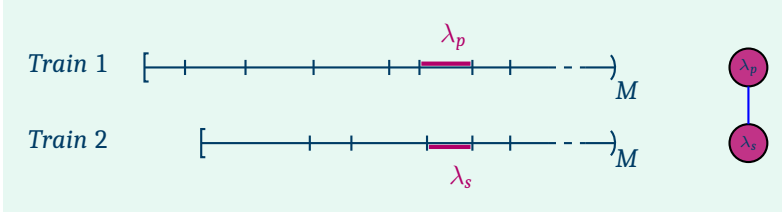


$$x_p = \begin{cases} 1 & \text{if train enters the track segment} \\ & \text{at some time in the interval } \lambda_p \\ 0 & \text{otherwise} \end{cases}$$

DDD-Incompatibility

Given two trains traversing the same track segment, two distinct intervals λ_p and λ_s are said DDD-incompatible if for **any** $t \in \lambda_p$ **and any** $t' \in \lambda_s$ we have:

$$[t, t + l) \cap [t', t' + l) \neq \emptyset$$



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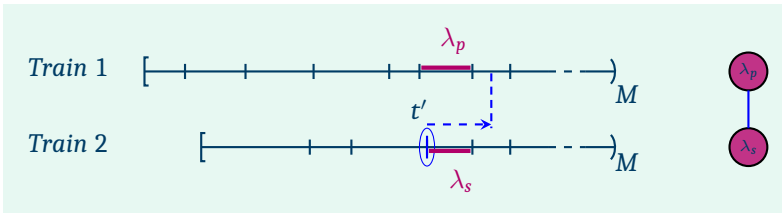

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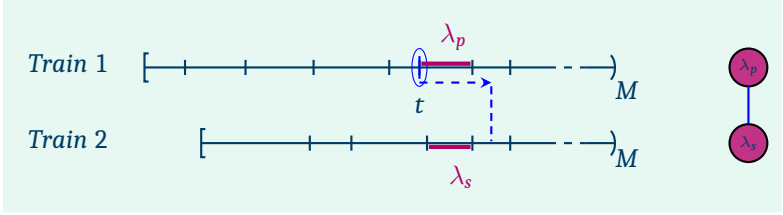

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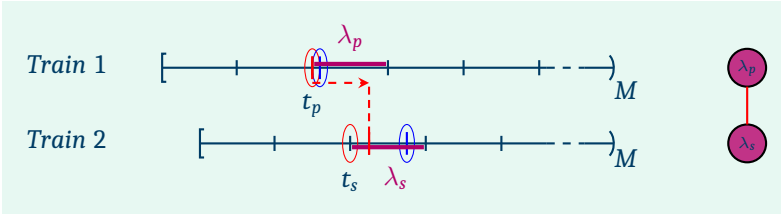
TI-Incompatibility vs DDD-incompatibility



Note:

Two intervals λ_p and λ_s can be

TI-incompatible and **not DDD-incompatible**

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We can obtain a schedule $t^* = \Phi(x^*)$,
t^* is a lower bound for the TRP

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DDD algorithm

To solve a scheduling problem, we can now:

1. Choose an initial set of partitions Λ.
2. **Solve** the Interval Assignment Problem (selecting one interval λ_p from each Λ^{ir}).
3. If the schedule using t_p (the first point in each interval) is **feasible**, `return`.
4. **Refine** the set of partitions Λ and `goto 2`.

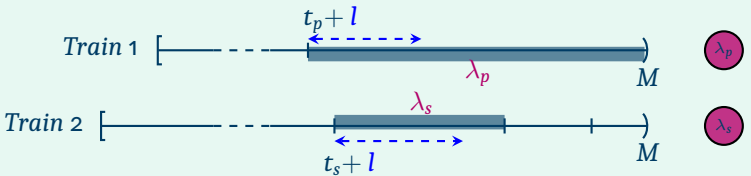
Initialize the IAP

The initial problem D_0 considers for each track segment traversed by each train, a single interval of the type:

$$\Lambda_0 = \{[\underline{t}, M), \text{ for each } i \in I \text{ and each } r \in R_i\}$$

Refine the IAP problem

Example of refinement for two non-DDD-incompatible intervals



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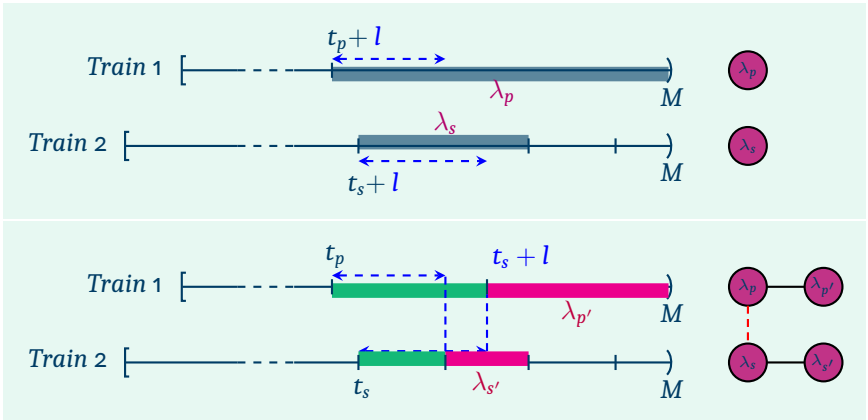

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DDD pros and cons

- **Advantages**:
  - Coarse discretization might suffice to find an optimal solution.
  - No pre-specified granularity – exact (continuous) solutions.
- **Challenges**:
  - Need to solve a *sequence* of MILPs.

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Computational experiments

- In our experience, running the **DDD algorithm** using a **MILP solver (Gurobi)** is not competitive with the big-M formulation.
- Row and column (variables and constraints) generation: MILP solvers typically **restart** completely.

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Computational experiments

- In our experience, running the **DDD algorithm** using a **MILP solver (Gurobi)** is not competitive with the big-M formulation.
- Row and column (variables and constraints) generation: MILP solvers typically **restart** completely.

- Two ways forward:
  1. Use a more incremental solver: **core-based MaxSAT**
  2. Use a custom branch-and-bound algorithm

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SAT solvers

- The Boolean satisfiability (SAT) problem asks whether there is an assignment to binary variables that satisfies a set of **clause** constraints:

$$x_1 + \ldots + x_k + (1 - x_{k+1}) + \ldots + (1 - x_n) \geq 1$$

- Very good open source solvers such as MiniSat and CaDiCaL.

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- Very good open source solvers such as MiniSat and CaDiCaL.
- Many applications in computer science, many based on **incremental** use – similar to **row and column generation**.
- Last 5 years has seen good progress also in MaxSAT, the optimization version of SAT – no “native” numbers, so typically small integer objectives.

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SAT solvers

Train scheduling DDD translates nicely to clause constraints

$\Rightarrow$ can be solved as a **MaxSAT problem**.

(we solved it using the RC2 algorithm)

Instances

The test set consists of **24 real-life instances** derived from two single-track railway networks of the Norwegian railroad.

	Line A	Line B
Number of instances	12	12
Number of routes	33	25
Avg Train	20	11
Avg Track per Train	19	15

We created an **additional 48 test instance**s by letting some trains take longer to travel tracks or wait longer in stations.

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Objective functions

We minimize the train **delays at their final destination** stations f considering a delay function of the type:

$$c(t^{if}) = \max(0, t^{if} - \underline{t}^{if})$$

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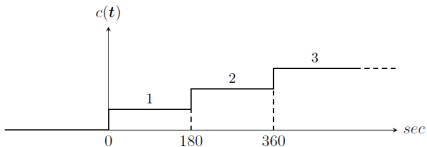
We tested stepwise functions with different number of steps:

1. **Linear rounded function**: $\Sigma_{t^{if} \in F} \lfloor c(t^{if})/Q \rfloor$
   where Q=180 is the time between stepwise increases
2. **3 steps function (0-3-6min)**:



3. **1 step function (0-5min)**.

Computational results

DDD-ALG and Big-M computation times (in ms) for different objective functions on the **10 hardest instance**s of our set.

	Linear rounded		3 steps		1 step	
Instance	Big-M	DDD-ALG	Big-M	DDD-ALG	Big-M	DDD-ALG
$\mathcal{I}_{11}^{AT}$	T/O	T/O	2541	453	689	221
$\mathcal{I}_{12}^{AT}$	T/O	T/O	2405	362	442	231
$\mathcal{I}_{12}^{AS}$	T/O	T/O	2080	380	565	172
$\mathcal{I}_{11}^{AS}$	T/O	T/O	917	335	566	198
$\mathcal{I}_{8}^{AS}$	115622	40404	1811	241	1188	126
$\mathcal{I}_{8}^{AT}$	37574	13416	875	247	254	191
$\mathcal{I}_{1}^{AS}$	1694	512	1161	178	393	144
$\mathcal{I}_{11}^{BO}$	1187	78	123	23	71	17
$\mathcal{I}_{2}^{AS}$	555	306	372	83	127	46
$\mathcal{I}_{8}^{AO}$	587	198	200	57	92	77

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Computational results

It turned out that:

- when using a **linear rounded objective** function the DDD-ALG is typically between **2x and 10x faster** than Big-M (the rounding makes a large difference to the DDD-ALG)

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- when using a **linear rounded objective** function the DDD-ALG is typically between **2x and 10x faster** than Big-M (the rounding makes a large difference to the DDD-ALG)
- with a **3 steps objective** function the computation **times are always much lower** (the DDD-ALG is faster than Big-M by 3x-10x on all instances)

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- with a **3 steps objective** function the computation **times are always much lower** (the DDD-ALG is faster than Big-M by 3x-10x on all instances)
- with a **1 step objective** function the DDD-ALG has computational times lower than 300 ms
- when using a **linear objective**, the DDD-ALG is **not** a **successful** approach (*general weakness with exact core-based MaxSAT solvers*)

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Ways forward

- Primal/dual approaches (heuristic optimization inside selected intervals).
- Similarities between DDD and continuous-time Multi-agent path finding (MAPF).
- Branch-and-price DDD (A. Helber and M. Lübbecke).
- Adapting the DDD paradigm to other scheduling problems.

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